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LETTER TO THE EDITOR

Eden growth on multifractal lattices

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Abstract. A modified Eden model has been investigated in which the growth probabilities are determined by a fractal measure on the underlying lattice. Both the spatial distribution and probability distribution of surface sites on the growing clusters have been investigated. For the case where the multifractal substrate is constructed using a 2×2 multiplicative generator with the probabilities $P_1=1$, $P_2=R$, $P_3=R^3$ and $P_4=R^3$, the spatial distribution of surface sites and the sites comprising the inner and outer hulls have a fractal geometry which can be described by dimensionalities which depend on R. For the total surface this dimensionality converges to a value of about 1.76 ± 0.01 as $R\to0$. For the inner and outer hulls the fractal dimensionality approaches a value of about 1.48 ± 0.02.

The Eden (1961) model for the growth of cell colonies is the most simple of the non-equilibrium growth models. Despite its apparent simplicity, a complete understanding of some aspects of this model is still in the process of emerging. For this reason it has been studied extensively in recent years and a variety of modifications have been explored. In the original version of the Eden model the growth process is started with a single occupied lattice site and unoccupied surface sites (unoccupied sites with one or more occupied nearest neighbours) are occupied randomly with probabilities which are proportional to the number of occupied nearest neighbours. This model leads to compact structures (Richardson 1973) as do variations on the Eden model in which the growth probabilities depend only on the local environment (Meakin 1983). The surface of the structures grown using this model are of considerable interest (Plischke and Racz 1984, Family and Vicsek 1985, Jullien and Botet 1985a, b, Freche *et al* 1985, Kardar *et al* 1986, Hirsch and Wolf 1986, Zabolitzky and Stauffer 1986, Meakin *et al* 1986, Stauffer and Zabolitzky 1986) and represent one aspect of this model which is not yet fully understood (particularly in higher dimensions). In addition, the effects of lattice anisotropy on the overall structure of Eden clusters is an area of active investigation (Dhar 1986, Freche *et al* 1985, Hirsch and Wolf 1985, Zabolitzky and Stauffer 1986, Meakin *et al* 1986).

Most of the work on Eden models has been carried out on a simplified version in which unoccupied surface sites are selected randomly and occupied with equal probability. The simplicity of these models allows very large structures ($\sim10^{10}$ sites) to be grown but uncertainties remain concerning the asymptotic scaling relationships which characterise their surfaces as a result of surprisingly large corrections to scaling. Recently Jullien and Botet (1985a, b) have introduced a third simple modification of the Eden model in which occupied surface sites are selected at random and a randomly selected unoccupied nearest-neighbour site is filled. This model seems to reduce the scaling corrections but uncertainties still remain concerning the surface structure associated with Eden models. Eden growth on fractal substrates (Mandelbrot 1982) has also been explored (Martin *et al* 1984). These and other aspects of Eden models have been reviewed recently by Herrmann (1986).

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Here we introduce a modification of the Eden model in which the growth probabilities are determined by a fractal measure associated with the lattice on which the growth process is occurring. The multifractal lattices used in the simulations are illustrated in figure 1. The generation of this type of fractal measure is discussed in the preceding letter (Meakin 1987a). Figure 1 illustrates the measure on a 128×128 ($2^7 \times 2^7$) square lattice at the seventh stage of iteration, using a generator of the type shown in figure 1(a) of Meakin (1987a), for the case $P_1 = 1$, $P_2 = 0.7$, $P_3 = 0.49$ and $P_4 = 0.343$. In the limit where the number of iterations, n, becomes infinite, the measure associated with each of the lattice sites (which has the form $P_1^i P_2^j P_3^k P_4^l$ with $i + j + k + l = n$) becomes a fractal measure or multifractal (Mandelbrot 1974, 1982, Halsey *et al* 1986a). The results reported here were obtained with multifractal lattices of the type $P_1 = 1$, $P_2 = R$, $P_3 = R^2$ and $P_4 = R^3$. Similar simulations have been carried out for other cases of the general type illustrated in figure 1. A more complete description of these substrate lattices can be found elsewhere (Meakin 1987a, b). The same type of multifractal constructions have been used to represent atmospheric turbulence (Mandelbrot 1974, 1982, Frisch *et al* 1978, Schertzer and Lovejoy 1983, Benzi *et al* 1984, Lovejoy and Schertzer 1986).

The Eden growth process starts with a single occupied lattice site and unoccupied surface sites are occupied randomly with probabilities which are proportional to the probability measure, μ, associated with them. The simulations were carried out on 1024×1024 ($2^{10} \times 2^{10}$) site square lattices (10 generations) and the growth process was continued until either 100 000 lattice sites were occupied or the edge of the lattice was reached by a cluster growing from the centre of the lattice. Figure 2(a) shows a 77 648 site cluster grown in this manner with the parameter R set to a value of 0.1 ($P_1 = 1$, $P_2 = 0.1$, $P_3 = 10^{-2}$, $P_4 = 10^{-3}$). Figure 2(b) shows all of the unoccupied sites which are adjacent (nearest neighbours) to an unoccupied site on the cluster. These are the potential growth sites. Figure 2(c) shows the 'inner' hull of the cluster. This hull consists of all of those occupied surface sites which can be reached from outside of the cluster by paths consisting of steps from the unoccupied sites to nearest-neighbour or next-nearest-neighbour unoccupied sites. For percolation clusters this hull has a fractal dimensionality which is close to or equal to $\frac{7}{4}$ (Voss 1984, Sapoval *et al* 1985, Ziff 1986, Saleur and Duplantier 1987, Coniglio *et al* 1987). Figure 2(d) shows the 'outer' hull of the cluster. This hull consists of all those unoccupied surface sites (potential growth sites) which can be reached from outside of the cluster by paths connecting unoccupied nearest neighbours only. For percolation clusters this hull has a fractal dimensionality which is equal to or close to $\frac{4}{3}$ (Grossman and Aharony 1986, Meakin and Family 1986). Using the model described above, a number of clusters were grown to investigate the quantitative aspects of the model (425 with $R = 0.8$, 68 with $R = 0.4$, 34 with $R = 0.2$ and 19 with $R = 0.1$). Figure 3 shows the two-point density correlation function $C(r)$ for the total surface (unoccupied sites), the internal hull and the external hull. In all cases a significant range of length scales is found over which these correlations have the power-law form characteristic of fractal structures:

$$C(r) \sim r^{-\alpha} \qquad D_\alpha = d - \alpha. \tag{1}$$

In equation (1) d is the Euclidean dimensionality of the lattice. It is also apparent from figure 3 that the correlation functions approach a limiting shape as R is reduced to smaller and smaller values. Table 1 summarises the results obtained for D_α using several values for R. These results suggest limiting ($R \to 0$) fractal dimensionalities of

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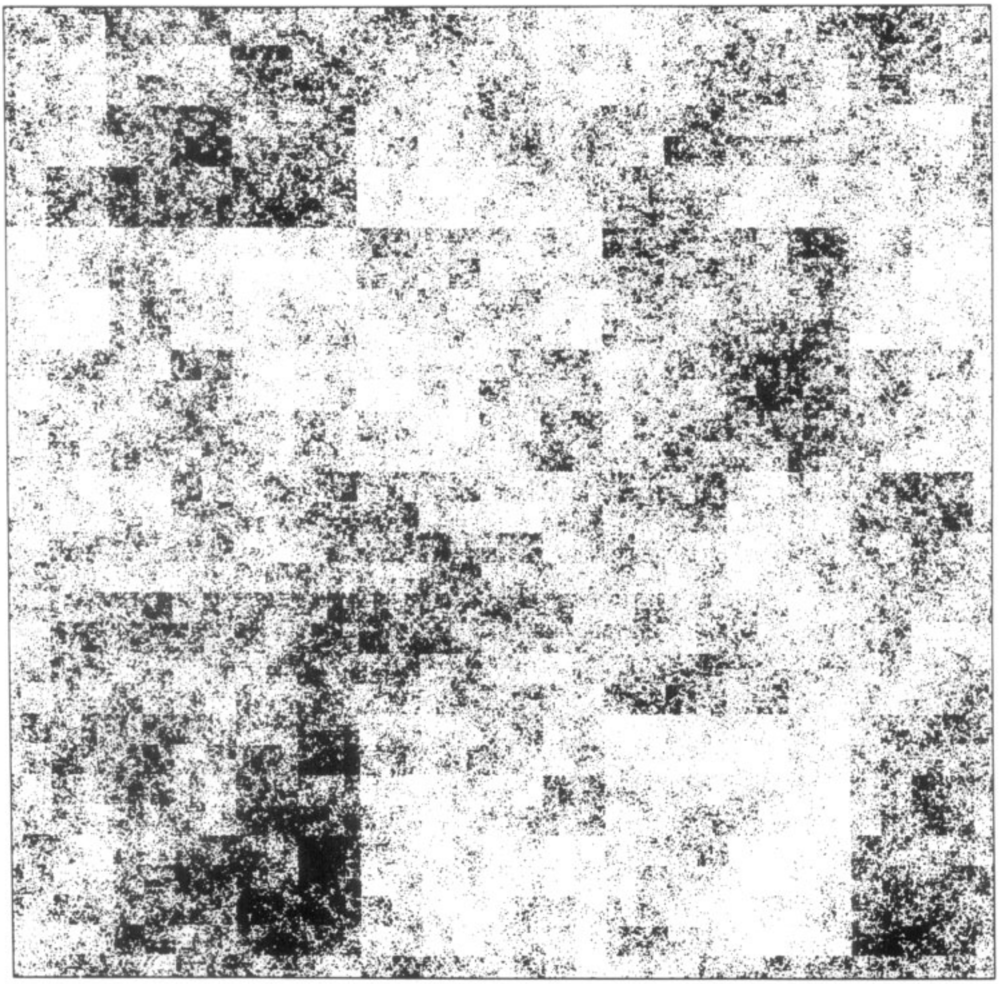

Figure 1. An illustration of the type of multifractal substrate used in this work. Repeated application of a generator of the type illustrated in figure 1(a) of Meakin (1987a) to each of the elements in the system generates a multifractal measure on the lattice. Here the density of randomly placed points at each of the lattice sites is proportional to the value of the measure at that site. The width of the figure is 128 lattice units.

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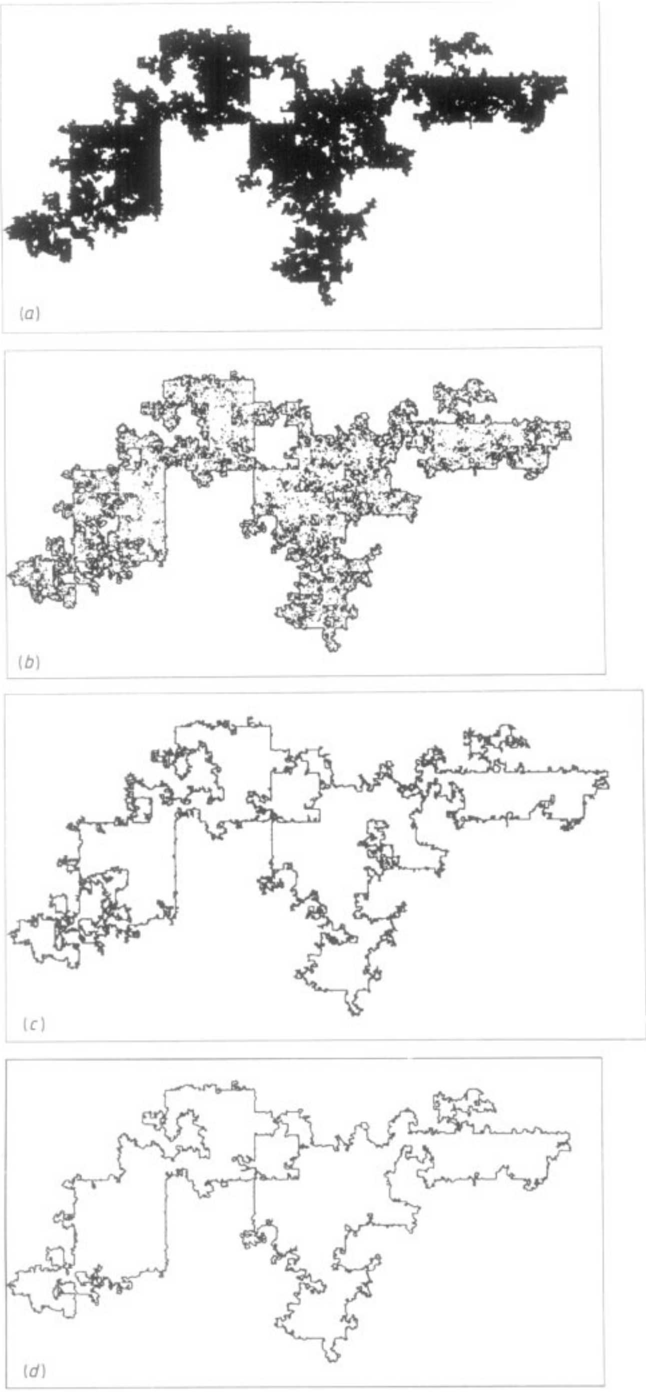

Figure 2. A cluster grown on the substrate with the generator $P_1 = 1$, $P_2 = R$, $P_3 = R^2$, $P_4 = R^3$ (1024×1024 lattice units or 10 generations). (*a*) shows the site occupied by the cluster, (*b*) shows all of the unoccupied surface sites, (*c*) and (*d*) show the sites associated with the inner and outer hulls respectively. The cluster contains 77 648 sites and the width is 850 lattice units.

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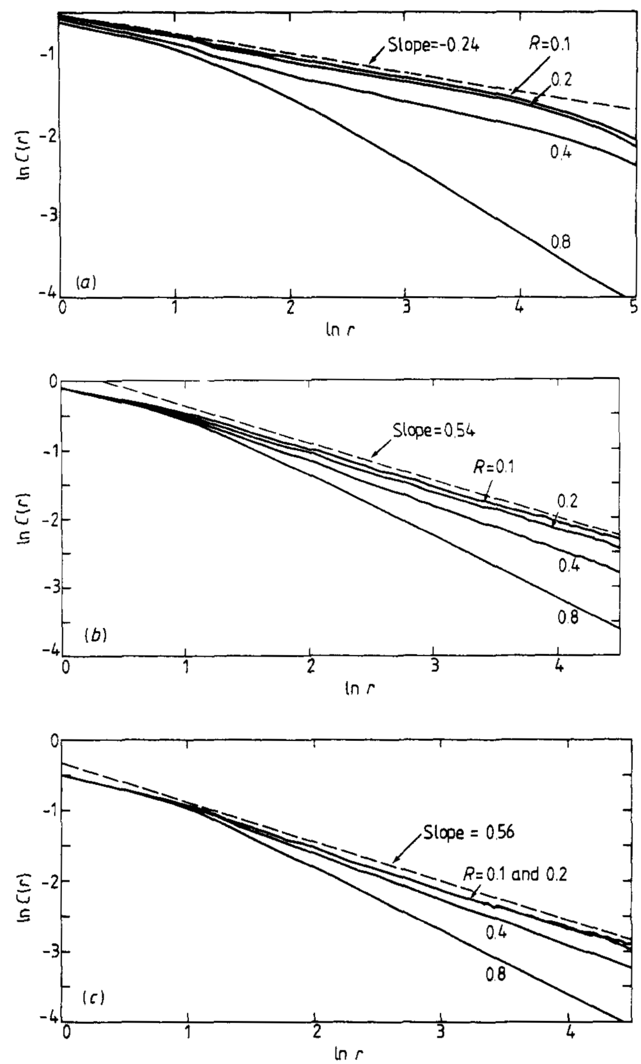

Figure 3. Density-density correlation functions obtained from Eden growth on multifractal substrates of the type $P_1 = 1$, $P_2 = R$, $P_3 = R^2$ and $P_4 = R^3$. (*a*) shows the correlation functions obtained from the total (unoccupied) surface, (*b*) and (*c*) show the correlation functions for the inner and outer hulls, respectively, obtained for four different values of $R = 0.1$, 0.2, 0.4 and 0.8.

1.75 ± 0.01 for the total surface and 1.48 ± 0.02 for both the inner and outer hulls. A direct simulation in the limit $R \to 0$ using 129 clusters led to values of 1.77 ± 0.01, 1.47 ± 0.01 and 1.43 ± 0.01 for the total surface, inner hull and outer hull, respectively. The measure associated with each of the lattice sites has the form R^m. Simulations can be carried out in the limit $R \to 0$ by including only those unoccupied surface sites with the smallest value of m in the list of growth sites during any particular stage in the growth process. The sites in the list are then selected with equal probability.

A subject of considerable current interest is the distribution of growth probabilities in aggregation models (see Meakin *et al* 1985, Halsey *et al* 1986b, for example). This distribution of growth probabilities was measured for the clusters grown using the

Table 1. Effective fractal dimensionalities, D_α, obtained from Eden growth on a multifractal lattice with the generator (see figure 1) given by $P_1 = 1$, $P_2 = R$, $P_3 = R^2$ and $P_4 = R^3$. The standard error obtained from the least squares fit is smaller than 0.002. However, systematic uncertainties are probably much larger. These results were obtained from the correlation functions shown in figure 3.

R	Total surface	Internal hull $5 \leq r \leq 100$	External hull $5 \leq r \leq 100$
0.8	1.116	0.103	1.103
0.4	1.674	1.352	1.354
0.2	1.753	1.448	1.437
0.1	1.755	1.471	1.466

model described above. The cluster growth process was stopped at 8 stages ($M = 1000$, 2000, 4000, 8000, 16 000, 32 000, 64 000 and 100 000 occupied sites) in order to measure the distribution of growth probabilities. Because of the finite size of the system and the way in which the multifractal substrate was constructed, there are only 31 possible values for the growth probability measure, μ, at a particular lattice site (R^0, $R^1, \ldots, R^{30}$). Figure 4 shows the fraction of sites, F_m, with associated probabilities R^m for each of the 31 values of m. This distribution is a discrete histogram, but each of the points in the histogram has been connected by a line to its neighbours to form a continuous curve. Also shown in figure 4 is the histogram for the substrate which (in the limit of a large number of generations) approaches a log-binomial distribution. Figure 4 shows that, as R becomes smaller, the distribution of growth probabilities becomes considerably narrower than the distribution of probabilities in the substrate. This is not surprising since sites with very high growth probabilities are strongly correlated with other sites of high growth probability and these regions will be filled rapidly once they become accessible to the growth process. Similarly the growth process will avoid regions with very low growth probabilities so that sites with low growth probabilities will rarely be found at the surface of the growing cluster.

In contrast to most growth models, in which the distribution of growth probabilities becomes broader as the clusters grow, in this case this distribution becomes narrower. However, this is a result of the fact that we have averaged the growth probabilities over all clusters. If the growth probabilities were normalised for each cluster and then averaged, we would see the distribution of normalised growth probabilities grow with increasing cluster size and could use this information to estimate the function $f(\alpha)$ describing the 'spectrum' of singularities in the growth probability measure (Halsey *et al* 1986a). The distribution of normalised growth probabilities is presently under investigation.

An interesting question which arises from these results is the dependence of the limiting ($R \to 0$) fractal dimensionalities of the total surface and the hulls on the form of the generator. In this case we have used the generator 1, R, R^2, R^3. Will other generators (1, S, S, S^2 or 1, 1, T, T, for example) give the same limiting values? Additional computer simulations are being carried out to answer this question.

It is reasonable to believe that there might be some connection between the $R \to 0$ limit of this model and percolation. However, the fractal dimensionalities which we have obtained for the total surface and hulls seem to be approaching limiting values as $R \to 0$ which are not the same as those found in percolation.

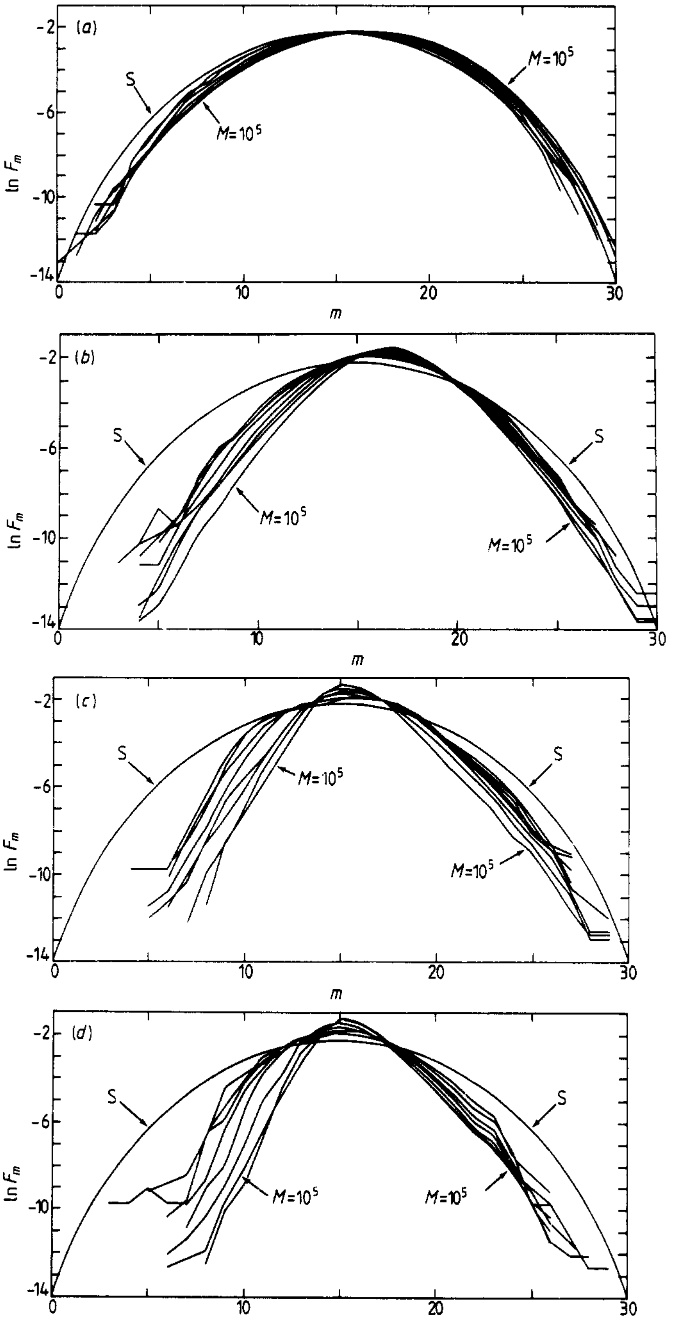

Figure 4. Growth probability histograms for clusters to size 1000, 2000, 4000, 8000, 16 000, 32 000, 64 000 and 100 000 occupied sites, for four different values of $R = (a)$ 0.8, (b) 0.4, (c) 0.2 and (d) 0.1. These results were obtained from a large number of clusters with random growth sites and are averaged over all of the clusters. Consequently the distribution of growth probabilities is the same as that for the substrate (smooth curves labelled S) for very small clusters and becomes narrower as high growth probability regions are filled and low probability regions are avoided. Here F_m is the fraction of sites which have associated with them probabilities of R^m.

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References

Benzi R, Paladin G, Parisi G and Vulpiani A 1984 *J. Phys. A: Math. Gen.* **17** 3521
Coniglio A, Jan N, Majid I and Stanley H E 1987 *Phys. Rev.* B **35** 3617
Dhar D 1986 *On Growth and Form: Fractal and Non-fractal Patterns in Physics. NATO ASI Ser.* ed H E Stanley and N Ostrowsky (Dordrecht: Martinus Nijhoff) pp 288-92
Eden M 1961 *Proc. 4th. Berkeley Symp. on Mathematical Statistics and Probability* vol 4, ed F Neyman (Berkeley: University of California Press) p 233
Family F and Vicsek T 1985 *J. Phys. A: Math. Gen.* **18** L75
Frisch U, Sulem P and Nelkin M 1978 *J. Fluid Mech.* **87** 719
Freche P, Stauffer D and Stanley H E 1985 *J. Phys. A: Math. Gen.* **18** L1163
Grossman T and Aharony A 1986 *J. Phys. A: Math. Gen.* **19** L745
Halsey T, Jensen M H, Kadanoff L, Procaccia I and Schraiman B I 1986a *Phys. Rev.* A **33** 1141
Halsey T C, Meakin P and Procaccia I 1986b *Phys. Rev. Lett.* **56** 854
Herrmann H J 1986 *Phys. Rep.* **136** 155
Hirsch R and Wolff D E 1986 *J. Phys. A: Math. Gen.* **19** L251
Jullien R and Botet R 1985a *Phys. Rev. Lett.* **54** 2055
—— 1985b *J. Phys. A: Math. Gen.* **18** 2279
Kardar M, Parisi G and Zhang Y C 1986 *Phys. Rev. Lett.* **56** 889
Lovejoy S and Schertzer D 1986 *Bull. Am. Meteor. Soc.* **67** 221
Mandelbrot B B 1974 *J. Fluid Mech.* **62** 331
—— 1982 *The Fractal Geometry of Nature* (San Francisco: Freeman)
Martin H, Vannimenus J and Nadal J P 1984 *Phys. Rev.* A **30** 3205
Meakin P 1983 *Phys. Rev.* B **28** 6718
—— 1987a *J. Phys. A: Math. Gen.* **20** L771
—— 1987b *Preprint*
Meakin P, Botet R and Jullien R 1986 *Europhys. Lett.* **1** 609
Meakin P and Family F 1986 unpublished
Meakin P, Stanley H E, Coniglio A and Witten T A 1985 *Phys. Rev.* A **32** 2354
Plischke M and Racz Z 1984 *Phys. Rev. Lett.* **53** 415
Richardson D 1973 *Proc. Camb. Phil. Soc.* **74** 515
Saleur H and Duplantier B 1987 *Preprint*
Sapoval B, Rosso M and Gouyet J F 1985 *J. Physique* **46** 2149
Schertzer D and Lovejoy S 1983 *Preprints, IUTAM Symp. on Turbulence and Chaotic Phenomena in Fluids, Kyoto, Japan* pp 141-4
Stauffer D and Zabolitzky J G 1986 *Phys. Rev. Lett.* **57** 1809
Voss R F 1984 *J. Phys. A: Math. Gen.* **17** L373
Zabolitzky J G and Stauffer D 1986 *Phys. Rev.* **34** 1523
Ziff R M 1986 *Phys. Rev. Lett.* **56** 545